MASSIVE GRAVITY IN ADS AND MINKOWSKI BACKGROUNDS

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Abstract

I review some interesting features of massive gravity in two maximally symmetric backgrounds: Anti de Sitter space and Minkowski space. While massive gravity in AdS can be seen as a spontaneously broken, UV safe theory, no such interpretation exists yet in the flat-space case. Here, I point out the problems encountered in trying to find such completion, and possible mechanisms to overcome them.

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1 Introduction

While doing theoretical research in gravity, both classical and quantum, standard or "super," at some stage we inevitably encounter a major contribution due to Stanley Deser. My experience is no exception: in [1], Boulware and Deser presented a comprehensive and in some way definitive study of massive gravity in four dimensions. Their analysis pointed out the true reason underlying the problems faced by a quantum theory of massive gravity. It has nothing to do with the incompleteness of Einstein gravity at high energy. Rather, it is a truly infrared problem. The problem is best explained using the ADM formalism [2], another major contribution to gravity due to Stanley.

In ADM, the physical, propagating degrees of freedom of massless gravity are the space metric, g_{ij} $i = 1, 2, 3$ and its conjugate momenta π^{ij}. The other components of the metric, $N = (-g^{00})^{-1/2}$ and $N^i = g^{i0}$, are nondynamical and appear linearly in the Einstein action. So, they act as Lagrange multipliers, enforcing 4 extra constraints. They, together with the 4 gauge invariances following from general covariance, remove 8 of the 6+6 degrees of freedom, leaving only 2 propagating degrees of freedom (2 generalized coordinates and 2 conjugate momenta). A Lorentz invariant mass term does not change the fact that N^i, N are nondynamical, but makes them appear nonlinearly in the action of massive gravity. So, their equations of motion do not produce any new constraint, and one ends up with 6 propagating degrees of freedom. One of them is always either a ghost or a tachyon. The only exception to this conclusion obtains when Einstein's action is modified by adding a Pauli-Fierz [3] mass term that, at quadratic order in the metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{mn\nu}$, reads

$$S_M = S_{Einstein} + \frac{M^2}{64\pi G} \int d^4x (h_{\mu\nu} h^{\mu\nu} - h^2). \tag{1}$$

In this action, the lapse (better $h_{00} \approx N^2 - 1$) appears linearly, so that it still acts as a multiplier, and does eliminate the unwanted sixth degree of freedom. On the other hand, Boulware and Deser showed that this property of the Pauli-Fierz mass term holds only in the quadratic approximation. In any Lorentz-invariant mass term, N does enter nonlinearly in the complete, interacting Lagrangian. Correspondingly, the full nonlinear theory propagates 6 degrees of freedom (and the Hamiltonian is unbounded below).

The results of Boulware and Deser mean that massive gravity cannot be a consistent quantum theory. The recent revival of interest in massive gravity, or, more generally, in long-distance modifications to gravity, does not contradict that. The question posed in recent years is not whether a quantum theory of massive gravity exists, that makes sense up to a very high (Planckian) energy. The question is instead whether massive gravity makes sense as a *low-energy effective field theory*, up to the shortest scale at which we have experimentally tested gravity. From this point of view, the UV cutoff of the

theory is not M_{Pl} but the (somewhat smaller!) scale $\sim (100\,\mu m)^{-1} \sim 10^{-3}\,eV$. Surprisingly, finding an effective theory that works up to such a small cutoff and is compatible with experiment is nevertheless difficult. Let us review the problems, starting with the case that is better understood theoretically, namely, massive gravity in AdS space.

2 Massive Gravity in AdS Space

This section is based on [4, 5].

Neither in flat space, nor in anti de Sitter space is the long-distance behavior of Einstein's gravity changed by coupling it to massive particles. The effect of massive particles is always encoded in local operators that do not give a mass term to the graviton, because of general covariance. On the other hand, in AdS space, the effect of massless particles is subtler. Let us take for instance a very simple case: a conformally coupled free scalar. Free means here that the scalar interacts only with gravity. By integrating out the scalar field, one gets a *nonlocal* action for the graviton, that can be written schematically as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) + W_{CFT}[g]. \tag{2}$$

Λ is the (negative) cosmological constant of the background, and $W_{CFT}[g]$ denotes the generating functional of the connected correlators of the CFT. Denote by $\bar{g}_{\mu\nu}$ the background, and expand the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. Then the linearized equations of motion obtained by varying the action in Eq. (2) are

$$L^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} + \Sigma^{\alpha\beta}_{\mu\nu}h_{\alpha\beta} = 0. \tag{3}$$

Here, $L^{\alpha\beta}_{\mu\nu}$ is the standard Einstein kinetic operator in AdS, and

$$\Sigma^{\alpha\beta}_{\mu\nu} = \left.\frac{\delta^2 W_{CFT}}{\delta g^{\mu\nu}\delta g_{\alpha\beta}}\right|_{g=h} = \langle T_{\mu\nu}T^{\alpha\beta}\rangle_{CFT}. \tag{4}$$

So, Σ is the two-point function of the stress-energy tensor in the CFT. Implicit in this notation is the fact that while the CFT is integrated out exactly, gravity is treated classically, i.e. all graviton loops are being ignored. This approximation makes sense for computing infrared quantities on a weakly curved background.

By a wise choice of counterterms, Σ can be made transverse traceless with respect to the background metric. By construction, the equations of motion are covariant. So, one can decompose the metric fluctuation into transverse-traceless (TT), longitudinal, and trace components, and choose the gauge $h_{\mu\nu} = h^{TT}_{\mu\nu} + \bar{g}_{\mu\nu}\phi$. In this gauge the equations of motion split into

$$\left[\frac{1}{32\pi G}(\Delta - 2\Lambda) + \Sigma(\Delta)\right]h^{TT}_{\mu\nu} = 0$$

$$(3\Delta - 4\Lambda)\phi = 0. \tag{5}$$

Here, Δ is the Lichnerowicz operator [6], a curved-space generalization of the Laplacian, that commutes with the covariant divergence and trace defined in terms of the background metric. The scalar operator $\Sigma(\Delta)$, computed at the pole of the propagator, gives the graviton square mass. In the Einstein theory, the pole is at $\Delta = 2\Lambda$, and $\Sigma(2\Lambda) = 0$. When the mass is smaller than $2|\Lambda|$, this prescription gives

$$M^2 \approx 32\pi G\Sigma(2\Lambda). \tag{6}$$

Now, even before doing any explicit computation, it is clear that this mass is parametrically smaller than the curvature radius of AdS, $L \equiv \sqrt{|\Lambda|/3}$. Indeed, Σ is computed by a correlator in the CFT, that depends on L but not on the Newton constant G. So, Σ can be at most $O(cL^{-4})$, where c is the central charge of the CFT; thus,

$$M \sim a\sqrt{c}L_{Planck}/L^2 \ll 1/L. \tag{7}$$

Here a is a number of order one. Of course, it could still be zero. The analysis performed in ref. [4] shows that a indeed vanishes when the conformally coupled scalar is given standard (reflecting) boundary conditions at the boundary of AdS. When the field is given more general boundary conditions, that allow for an energy flow into and out of the AdS space, then a is nonzero.

More precisely, the representation theory of the isometry group of AdS_4, $SO(2,3)$, shows that a conformally coupled scalar can belong to two representations, called $D(1,0)$ and $D(2,0)$, respectively (see [4] or [7] for notations and further results). When reflecting boundary conditions are given, then the scalar belongs to either $D(1,0)$ or $D(2,0)$ [8]. In general, the scalar field may be a linear combination of modes belonging to both representations. In the case that the linear combination is the same for all modes, then the scalar propagator is

$$\Delta(x,y) = \alpha\Delta^1(x,y) + \beta\Delta^2(x,y), \qquad \alpha + \beta = 1. \tag{8}$$

Here $\Delta^E(x,y)$ is the propagator for modes in the $D(E,0)$ irrep. Transparent boundary conditions, first proposed in [9], and natural from the point of view of the holographic AdS/CFT duality [10, 4], are $\alpha = \beta = 1/2$. The result of [4, 5] is

$$\Sigma(2\Lambda) = \alpha\beta\frac{G}{5\pi L^4}. \tag{9}$$

So, the graviton mass vanishes for standard boundary conditions, but it can be nonzero for nonstandard ones. Physically, the Higgs field (a vector belonging to $D(4,1)$ [4], in this case) is a composite field, a "bound state" of size L.

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In conclusion, in AdS one can give a (tiny) mass to the graviton by a Higgs mechanism. The Higgs is a composite vector of size L, and $1/L$ is the cutoff for the effective theory of massive gravity. Above that energy, the correct description is in terms of ordinary gravity plus all the degrees of freedom of the CFT.

3 Minkowski Space

No analog of the Higgs-like mechanism described in the previous section is available in Minkowski space. The best one can do is to introduce the appropriate Goldstone fields needed to make massive gravity explicitly covariant under general coordinate transformations. This can be done at the full nonlinear level [11]. In the Goldstone boson language, the disease first noticed by Boulware and Deser, that the lapse starts propagating at nonlinear level, manifests itself in a different guise: massive gravity becomes strongly interacting at an extremely small energy scale:

$$E \sim (M_{Pl} M^4)^{1/5}, \qquad \text{or,} \qquad E \sim (M_{Pl} M^2)^{1/3}. \tag{10}$$

The first scale holds for the standard PF mass term, while the second is the best that can be achieved by judiciously improving the action by adding appropriately chosen higher-order operators [11]. For a graviton of Compton wavelength $M^{-1} \sim 10^{28} \, cm$, the cutoff length corresponding to the highest energy cutoff in Eq. (10) is $O(1000 \, km)$! This is the scale below which massive gravity becomes strongly interacting and essentially uncontrollable within perturbation theory.

The existence of such very low scale is not confined to the theory of a single massive graviton. A similar bound also exists in the DGP model [12]. The DGP model is the first ghost-free example of a mechanism in which gravity can be localized on a 4d brane in a space of infinite transverse volume. It describes a theory where 4d general covariance is unbroken, but the graviton is a metastable state. Its main property is that, on the 4d brane, gravity looks 4d at short distance, while it weakens at large distance. Interesting cosmological applications of this scenario have been proposed, for instance in [13].

The model can be described by the action

$$S_{\text{DGP}} = \frac{1}{16\pi G_5} \int_{\mathcal{M}} d^5 x \sqrt{-g} \, R(g) \tag{11}$$

$$+ \int_{\partial \mathcal{M}} d^4 x \sqrt{-\gamma} \left[-\frac{1}{8\pi G_5} K(\gamma) + \frac{1}{16\pi G} R(\gamma) \right], \tag{12}$$

where $\mathcal{M}$ is a 5d manifold with boundary $\partial \mathcal{M}$, g is the 5d metric, γ is the 4d induced metric on the boundary, and K is the extrinsic curvature. The DGP model is closely related to massive gravity. In

fact, the brane-to-brane graviton propagator can be written as

$$D^{\text{DGP}}_{\mu\nu\,\rho\sigma}(p) = D^{\text{massive}}_{\mu\nu\,\rho\sigma}(p, |p|/L), \tag{13}$$

where $D^{\text{massive}}_{\mu\nu,\lambda\rho}(p, m^2)$ is the propagator for 4d massive gravity, and the ratio of the two Newton constants, $L = G_5/G$ defines the transition length from standard 4d gravity to the 5d behavior.

The analysis of [14] (see also [15]) shows that the DGP model becomes perturbatively strongly coupled at a scale $E = (M_{Pl}L^{-2})^{1/3}$. This is what one would get by naively substituting the "running mass" $|p|/L$ into the first of the bounds in Eq. (10). Ref. [14] also shows that this problem cannot be cured by adding local counterterms to the action Eq. (12).

So, strong coupling at an unacceptably low scale seems an ubiquitous problem plaguing IR-modified gravity. It could be resolved, perhaps, by a nonperturbative re-summation of Feynman diagrams, or by introducing other degrees of freedom that change the theory before it becomes strongly coupled. In the rest of this paper, I will briefly discuss the second possibility: the only one that would give us full computational control of the theory.

For simplicity, I will discuss massive gravity [1]. The strong coupling problem stems from the breakdown of the linearized approximation. This happens at a lower than expected scale because the linearized graviton fluctuation generated by a conserved source with stress energy tensor $T_{\mu\nu}$ is (in momentum space):

$$
\begin{aligned}
h_{\mu\nu}(p) &= \bar{h}_{\mu\nu}(p) + p_\mu p_\nu \Psi(p), \\
\bar{h}_{\mu\nu} &= \frac{8\pi G}{p^2 + M^2}\left[T_{\mu\nu}(p) - \frac{1}{3}T_{\mu\nu}(p)\right], \qquad \Psi(p) = -\frac{1}{3M^2}\frac{8\pi G}{p^2 + M^2}T^\mu_\mu(p).
\end{aligned}
\tag{14}
$$

The first term, $\bar{h}_{\mu\nu}$ is well-behaved in the limit $M \to 0$; the factor $-1/3$ in its trace component is the origin of the famous van Dam-Veltman-Zakharov discontinuity [18], and it also ensures that the only propagating degrees of freedom are the 5 physical polarizations of a massive spin-2 field. On the other hand, Ψ diverges in the massless limit. At liner order, this infrared divergence is harmless, since Ψ is a gauge mode: it vanishes in the one-graviton scattering amplitude, when $h_{\mu\nu}(p)$ is contracted with a conserved source. At the next order, though, it contributes an amplitude that may dominate over the linear term. For a point-like source of mass $\mathcal{M}$, inspection of Eq. (14) shows that this happens at distances $r = O(G\mathcal{M}/M^4)$. The length scale $(M^4 M_{Pl})^{-1/5}$ is attained for $\mathcal{M} = M_{Pl}$.

Can we modify massive gravity in such a way as to keep the same one-graviton amplitude in between conserved sources as given by Eq. (14)? The answer is: yes, by modifying the graviton's *kinetic* term.

Let us add to the linearized action S_M in Eq. (1) the term

$$S_A = \frac{A}{32\pi G}\int d^4x\, h(\partial_\mu\partial_\nu h^{\mu\nu} - \Box h), \tag{15}$$

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[1]Refs. [16, 17] are a first attempt to studying these issues in the DGP model.

where A is an arbitrary constant.

Decompose next the metric into its transverse-traceless, longitudinal, and trace components:

$$h_{\mu\nu} = h_{\mu\nu}^{TT} + \eta_{\mu\nu}\Phi + \partial_\mu\partial_\nu\Psi + \partial_{(\mu}A_{\nu)}^T, \qquad \partial^\mu A_\mu^T = 0. \tag{16}$$

By computing the double divergence of the equations of motion $(\delta(S_M + S_A)/\delta h_{\mu\nu} - T_{\mu\nu} = 0)$ we get

$$(M^2\Box + A\Box^2)\Phi = \partial_\mu\partial_\nu T^{\mu\nu}. \tag{17}$$

So, even when $A \neq 0$, Φ still obeys a homogeneous equation when $T_{\mu\nu}$ is conserved, and so it can be set to zero when computing the field generated by a localized source. This is the key property that guarantees that only 5 physical polarizations propagate in the one-graviton scattering amplitude. The unphysical gauge mode Ψ changes. It becomes

$$\Psi(p) = -\frac{1}{3(M^2 + Ap^2)}\frac{8\pi G}{p^2 + M^2}T_\mu^\mu(p). \tag{18}$$

For all $p^2 \neq 0$, this mode can be made arbitrarily small in the limit $A \to \infty$ [2]. Formally, the limit $A \to \infty$ eliminates the dangerous mode that triggers the breakdown of the linear approximation. Moreover, when expanding the metric as in Eq. (16), the change in the kinetic term can be easily interpreted as giving a very large kinetic term to the unwanted modes Φ and Ψ, that, therefore, decouple.

This argument is of course still hand-waving and it would be interesting to make it sounder. One objection that can be raised against it is that it makes $h_{\mu\nu}$ propagate 7 degrees of freedom, two of which are ghosts, instead of the physical 5. So, for any finite A, the theory is unstable unless the ghosts decouple from all physical amplitudes, as the do at linear order.

The existence of two extra degrees of freedom is another simple application of the methods devised by Boulware and Deser [1]. By writing the action in terms of the 3d metric, linearized lapse h_{00}, and shift h_{0i}, we see that S_A is

$$S_A = \frac{A}{16\pi G}\int d^4x(h_{ii} - h_{00})\partial_i\partial_0 h_{i0} + ... \tag{19}$$

So, $\partial_i h_{i0}$ and $h_{ii} - h_{00}$ become a new pair of (propagating) canonical variables. Now the total number of degrees of freedom is thus 6 (coming from h_{ij}) plus 1. The new degree of freedom is a boson with first-order action in the time derivative, so its energy is unbounded below.

Whether massive gravity –or DGP, where a more sophisticated version of this mechanism may be at work, according to the analysis of [16]– can be made perturbatively stable and calculable is

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[2]Static sources and quantum loop computations both require to use Euclidean momenta, so this condition is generic.

yet to be proved, many years after the groundbreaking investigations of Boulware and Deser. The problem is still tantalizing, so much so that it may be fit to conclude this review by mentioning that another intriguing route to solve the strong coupling problems of gravity has been recently opened: by explicitly breaking Lorentz invariance, gravity can be made finite-range, consistent with existing data, and weakly coupled down to distances $O(100\,\mu m)$. [19, 20, 21]

Acknowledgments

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